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A parsimonious quasi-diagonals-parameter symmetry model applied to unaided vision data

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Summary

For a 4×4 contingency table with ordered categories, this note proposes a parsimonious case of the quasi-diagonals-parameter symmetry model in Tomizawa (1990). The proposed model is an extension of the quasi-symmetry model. It is applied to unaided vision data of British women.

1. Introduction

Consider the 4×4 contingency table having ordered categories with cell probabilities (p_{ij}). Tomizawa (1990) considered the quasi-diagonals-parameter symmetry (QDPS) model defined by

$$p_{ij}=\begin{cases}\delta_{i-j}\alpha_j\psi_{ij} & (i>j),\\ \alpha_j\psi_{ij} & (i\le j),\end{cases}$$

where $\psi_{ij}=\psi_{ji}$ and $\alpha_1=\delta_1=1$ without loss of generality. [Note that this model was defined for the $R\times R$ table in Tomizawa (1990).] A special case of this model with $\delta_2=\delta_3=1\ (=\delta_1)$ is Caussinus' (1965) quasi-symmetry (QS) model. For the 2×2 sub-tables formed from rows i and j ($>i$) and columns s and t ($>s$) in the 4×4 table, let $\theta_{(i<j;s<t)}$ denote the corresponding odds ratio (i.e., a measure of association between rows and columns) defined as $\theta_{(i<j;s<t)}=(p_{is}p_{jt})/(p_{js}p_{it})$ for $1\le i<j\le 4;\ 1\le s<t\le 4$. Also let $\Omega_{(i<j;\,s<t)}=\theta_{(i<j;\,s<t)}/\theta_{(s<t;i<j)}$ [being the ratio of symmetric odds ratios with respect to the main diagonal of the table]. The QDPS model is equivalent to

$$\Omega_{(1<2;\,2<3)}=\Omega_{(2<3;\,3<4)}\quad(=\delta_2). \tag{1.1}$$

Key words: Quasi-symmetry model, square contingency table, symmetric (asymmetric) association

The QS model is equivalent to

$$\begin{cases} \Omega_{(1<2;\,2<3)} = \Omega_{(2<3;\,3<4)} = 1 & (= \delta_2), \\ \Omega_{(1<3;\,2<4)} = 1 & (= \delta_3). \end{cases} \tag{1.2}$$

[Goodman (1979) referred to the QS model (1.2) as the *symmetric* association model, and Tomizawa (1990) referred to the QDPS model (1.1) as the diagonals-parameter *asymmetric* association model. Note that the word "*asymmetric*" means that two odds ratios (being on the symmetric positions with respect to the diagonal), e.g., $\theta_{(i<j;\,s<t)}$ and $\theta_{(s<t;\,i<j)}$, are not equal.] Tomizawa (1990) also considered the parsimonious QDPS model with $\delta_2 = 1$, being equivalent to

$$\Omega_{(1<2;\,2<3)} = \Omega_{(2<3;\,3<4)} = 1 \quad (= \delta_2).$$

Table 1

Unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946; from Stuart (1955). (The values in parentheses are the maximum likelihood estimates of expected frequencies under the parsimonious QDPS model with $\delta_3 = 1$.)

Right eye grade		Left eye grade: Best (1)	Second (2)	Third (3)	Worst (4)	Total
Best	(1)	1520 (1520.0)	266 (270.6)	124 (122.0)	66 (63.4)	1976
Second	(2)	234 (229.4)	1512 (1512.0)	432 (434.6)	78 (80.0)	2256
Third	(3)	117 (119.0)	362 (359.4)	1772 (1772.0)	205 (205.6)	2456
Worst	(4)	36 (38.6)	82 (80.0)	179 (178.4)	492 (492.0)	789
Total		1907	2222	2507	841	7477

Consider the data in Table 1. Assume that the observations have a multinomial distribution. As described in Tomizawa (1990), each of the QDPS model, the QDPS model with $\delta_2 = 1$, and the QS model fits these data well (see Table 2a). In addition, according to the tests (with the 0.05 level) based on the difference between the likelihood ratio chi-squared values (G^2 values) for two nested models (see Table 2b), the QDPS model is preferable to both the QDPS model with $\delta_2 = 1$ and the QS model. Under the QDPS model applied to these data, the maximum likelihood estimates (MLEs) of δ_2 and δ_3 are $\hat{\delta}_2 = 1.342$ and $\hat{\delta}_3 = 0.843$. We now see that the $\hat{\delta}_3$ is somewhat closer to 1. So we are interested in applying the parsimonious QDPS model with $\delta_3 = 1$ (instead of $\delta_2 = 1$) to these data.

The purpose of this note is (i) to propose the parsimonious QDPS model with $\delta_3 = 1$ for the 4×4 table [which was not considered in Tomizawa (1990)], and (ii) to apply it to the data in Table 1.

2. A parsimonious QDPS model

For the 4x4 table, consider a model defined as

$$p_{ij} = \begin{cases} \delta_2 \alpha_j \psi_{ij} & (i-j=2), \\ \alpha_j \psi_{ij} & (i-j \neq 2), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$ and $\alpha_1 = 1$. This is the parsimonious QDPS model with $\delta_3 = 1$. This model is also equivalent to

$$\begin{cases} \Omega_{(1<2;\,2<3)} = \Omega_{(2<3;\,3<4)} & (= \delta_2), \\ \Omega_{(1<3;\,2<4)} = 1 & (= \delta_3). \end{cases} \tag{2.1}$$

Condition (2.1) indicates that there is both *symmetric* association and *asymmetric* association in the table [though the QS model indicates that there is a *symmetric* association in the table and the QDPS model indicates that there is an *asymmetric* association in the table].

Since the models considered here are log-linear models, the MLEs of expected frequencies under each model can be easily obtained using an iterative procedure, for example, the general iterative procedure for log-linear models of Darroch and Ratcliff (1972). We shall not go into these details here.

3. Analysis of unaided vision data

When the parsimonious QDPS model with $\delta_3 = 1$ is applied to the data in Table 1, this model fits very well (see Table 2a). In addition, we can acccept (with the 0.05 level) the hypothesis that the QDPS model with $\delta_3 = 1$ holds under the assumption that the QDPS model holds true (i.e., the hypothesis that $\delta_3 = 1$ under the assumption), according to the test based on the difference between the corresponding G^2 values (see Table 2b). Thus the parsimonious QDPS model with $\delta_3 = 1$ would be preferable to the original QDPS model for these data. Moreover, according to a similar test (see Table 2b), the parisimonious QDPS model with $\delta_3 = 1$ would be preferable to the QS model for these data.

Under the parsimonious QDPS model with $\delta_3 = 1$ applied to these data, the MLE of δ_2 is $\hat{\delta}_2 = 1.393$. Therefore this model provides that from (2.1), (i) if the odds that a women's *left* eye grade is i+1 instead of i (i=1,2) is estimated to be $\hat{\theta}_{(i+1<i+2;\, i<i+1)}$ times higher when the women's *right* eye grade is i+2 rather than when it is i+1, then the odds that a women's *right* eye grade is i+1 instead of i is estimated to be $1.393 \times \hat{\theta}_{(i+1<i+2;\, i<i+1)}$ times higher when the women's *left* eye grade is i+2 rather than when it is i+1; and (ii) if the odds that a women's

Table 2
Likelihood ratio chi-squared (G^2) values for models applied to data from Table 1
(a) G^2 values

Models	Degrees of freedom (df)	G^2	P-value
QS	3	7.27	0.064
QDPS with $\delta_2 = 1$	2	4.64	0.098
QDPS with $\delta_3 = 1$	2	0.66	0.719
QDPS	1	0.22	0.639

(b) Difference between G^2 values [Model (1) − Model (2)]

Model (1)	Model (2)	Difference between df	Difference between G^2	P-value
QS	QDPS	2	7.05	0.029
QS	QDPS with $\delta_2 = 1$	1	2.63	0.105
QS	QDPS with $\delta_3 = 1$	1	6.61	0.010
QDPS with $\delta_2 = 1$	QDPS	1	4.42	0.036
QDPS with $\delta_3 = 1$	QDPS	1	0.44	0.507

left eye grade is 3 instead of 1 is estimated to be $\hat{\theta}_{(2<4;\,1<3)}$ times higher when the women's *right* eye grade is 4 rather than when it is 2, then the odds that a women's *right* eye grade is 3 instead of 1 is estimated to be *identically* $\hat{\theta}_{(2<4;1<3)}$ times higher when the women's *left* eye grade is 4 rather than when it is 2.

The above interpretations (i) and (ii) show that there is both *symmetric* association and *asymmetric* association (rather than a symmetric association and also rather than an asymmetric association) for the data in Table 1.

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REFERENCES

Caussinus, H. (1965). Contribution à l'analyse statistique des tableaux de corrélation. *Annales de la Faculté des Sciences de l'Université de Toulouse* 29, 77-182.

Darroch, J.N. and Ratcliff, D. (1972). Generalized iterative scaling for log-linear models. *Annals of Mathematical Statistics* 43, 1470-1480.

Goodman, L.A. (1979). Simple models for the analysis of association in cross-classification having ordered categories. *Journal of the American Statistical Association* 74, 537-552.

Stuart, A. (1955). A test for homogeneity of the marginal distributions in a two-way classification. *Biometrika* 42, 412-416.

Tomizawa, S. (1990). Quasi-diagonals-parameter symmetry model for square contingency tables with ordered categories. *Calcutta Statistical Association Bulletin* 39, 53-61.

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